One-loop integrals at finite temperature

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Abstract
A method devised by the author is used to calculate analytical expressions for one-loop integrals at finite temperature. The regularization of the integrals yields expressions which have a non-polynomial nature in the inverse temperature. A comparison with previously published results is presented and the advantages of the present technique are discussed.

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1. Introduction

In this paper we present a new technique which allows us to evaluate to a high degree of precision one-loop integrals occurring in finite-temperature problems in quantum field theory. Such integrals typically involve series, due to summation over the the Matsubara frequencies, $\omega_n = 2\pi n T$, with $n = 0, \pm 1, \ldots$ [1], which cannot be done analytically.

The method that we propose here is derived from the powerful ideas of the linear delta expansion (LDE) and of variational perturbation theory (VPT) (see [2] and references therein); we briefly sketch how such methods work.

First one interpolates a non-perturbative Lagrangian (Hamiltonian) $\mathcal{L}$ with a solvable one $\mathcal{L}_0$, which depends on one or more arbitrary parameters; the interpolated Lagrangian is then split into a 'leading' term, chosen to be $\mathcal{L}_0$ itself and into a 'perturbation', $\mathcal{L} - \mathcal{L}_0$. Although the 'perturbative' term will not be *a priori* small, a perturbative expansion is carried out and physical quantities are calculated. To finite order in perturbation theory, one thus obtains an expression which contains an artificial dependence upon the arbitrary parameters: however, if the expansion were to be carried out to all orders such dependence would have to cancel out provided that the perturbative series converges. In order to minimize such dependence, one applies the principle of minimal sensitivity (PMS) [3], which selects the value of the arbitrary parameter for which the physical quantity is less sensitive to changes in the parameter itself. The parameter determined in this way is normally such to make $\mathcal{L} - \mathcal{L}_0$ a perturbation and depends on the 'natural' parameters present in $\mathcal{L}$. For this reason, the expansion obtained by

using this method does not provide, to a finite order, a polynomial in the parameters of the model, as would be the case for a genuinely perturbative technique.

The ideas behind the LDE and the VPT are really very general and have been successful in dealing with a variety of problems of quite different nature, ranging from quantum field theory to classical and quantum mechanics. In this paper, in particular, we pursue the application of some results which were recently obtained by the author [4]: in that paper the author developed a method to accelerate the convergence of a certain class of mathematical series (including the Riemann and Hurwitz zeta functions) and proved that the new series obtained in such a way converge exponentially to the correct results. As we will see in this paper, some of the series treated in [4] are relevant for the evaluation of one-loop integrals at finite temperature and will be used to obtain arbitrarily precise analytical approximations to such integrals which are not polynomials in the inverse temperature.

The paper is organized as follows: in section 2 we review some of the results of [4] and explain the method; in section 3 we apply the results of section 2 to the calculation of one-loop integrals at finite temperature and compare them with the results in the literature [5–8]; finally in section 4 we draw our conclusions and discuss further possible applications of this method.

2. The method

In this section we describe the method of [4], which allows one to accelerate the convergence of certain series. One of the applications which was discussed in that paper is to the zeta-like function defined as

$$\zeta(u, s, \xi) \equiv \sum_{n=0}^{\infty} \frac{1}{(n^u + \xi)^s}, \tag{1}$$

where s, u and ξ are real parameters. This series converges for $us > 1$. The special cases obtained by taking $(u, \xi) = (1, 1)$ or $u = 1$ correspond to the Riemann and Hurwitz zeta functions, respectively. Further examples of application of this method are contained in [4] and the interested reader should refer to that paper for further reading.

We write

$$\zeta(u, s, \xi) = \frac{1}{\xi^s} + \sum_{n=1}^{\infty} \frac{1}{(n^u + \xi)^s} \tag{2}$$

and by simple algebra we convert it to the form

$$\zeta(u, s, \xi) = \frac{1}{\xi^s} + \sum_{n=1}^{\infty} \frac{1}{n^{su}} \frac{1}{(1 + \lambda^2)^s} \frac{1}{(1 + \Delta(n))^s}, \tag{3}$$

having introduced the definition

$$\Delta(n) \equiv \frac{\xi/n^u - \lambda^2}{1 + \lambda^2}. \tag{4}$$

We stress that equation (3) is still an identity and that no approximation has been made so far. λ here is an arbitrary parameter which was introduced 'ad hoc': this procedure is typical of the linear delta expansion (LDE) and of variational perturbation technique (VPT) [2].

Clearly, when the condition $|\Delta(n)| < 1$ is met, one can expand equation (3) in powers of Δ; such a condition requires that $\left|\frac{\xi - \lambda^2}{1 + \lambda^2}\right| < 1$, i.e. that $\lambda^2 > \frac{\xi - 1}{2}$ and $\xi > -1$.

We can therefore write

$$\frac{1}{(1 + \Delta(n))^s} = \sum_{k=0}^{\infty} \frac{\Gamma(k + s)}{\Gamma(s)k!} (-\Delta(n))^k \tag{5}$$

and convert the series (3) to the equivalent series:

$$\zeta(u, s, \xi) = \frac{1}{\xi^s} + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{\Gamma(k+s)}{\Gamma(s)k!} \sum_{j=0}^{k} \binom{k}{j} \frac{\lambda^{2(k-j)}}{(1+\lambda^2)^{s+k}} \frac{(-\xi)^j}{n^{u(s+j)}}. \tag{6}$$

By interchanging the sums and performing the sum over n, we finally obtain the expression

$$\zeta(u, s, \xi) = \frac{1}{\xi^s} + \sum_{k=0}^{\infty} \frac{\Gamma(k+s)}{\Gamma(s)} \sum_{j=0}^{k} \frac{(-\xi)^j}{j!(k-j)!} \frac{\lambda^{2(k-j)}}{(1+\lambda^2)^{s+k}} \zeta(u(s+j)). \tag{7}$$

We stress that equation (7) is still *exact* for $\lambda^2 > \frac{\xi-1}{2}$ and $\xi > -1$ and that it converges exponentially. Actually, this equation describes an entire family of series, all converging to the same function, although with different rates of convergence. When the sum in equation (7) is truncated to a finite value, a fictitious dependence on the arbitrary parameter λ is generated: in order to minimize such dependence and to obtain a series with an optimal rate of convergence, one can apply the principle of minimal sensitivity (PMS) [3], corresponding to selecting the value of λ for which the derivative of the partial sum with respect to λ vanishes.

To leading order, i.e. considering only the first two terms in the series, one obtains the result

$$\lambda_{\text{PMS}}^{(1)} = \sqrt{\xi} \sqrt{\frac{\zeta(u(1+s))}{\zeta(su)}}, \tag{8}$$

which can then be used inside equation (7). Note that equation (8) implies that $\xi \geqslant 0$. The precision of these formulas has been investigated in [4], showing that extremely precise results are in general obtained already working to low orders.

A final remark is in order: in principle, one would like to calculate λ_{PMS} to high orders; while this is certainly possible, it is not the best option to follow since the high-order expressions for λ_{PMS} are normally very involved. On the other hand, one can use $\lambda_{\text{PMS}}^{(1)}$ *to all orders* as long as it fulfils the condition $\left(\lambda_{\text{PMS}}^{(1)}\right)^2 > (\xi - 1)/2$. For example, if one takes $u = 2$ and $s = 3/2$, a case which we will consider later in this paper, this condition is *always* met. Of course, the price that one pays with this choice is a slightly smaller convergence rate, which however is definitely shadowed by the analytical simplicity.

3. Applications: one-loop integrals at finite temperature

We will follow the notation set in [6] and write the expression for a general one-loop integral at finite temperature in the form

$$J(m, a, b) = T\mu^{2\epsilon} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \int \frac{d^D k}{(2\pi)^D} \frac{(k^2)^a}{\left[k^2 + \omega_n^2 + m^2\right]^b},$$

μ being the scale brought in by dimensional regularization, and a and b being integers ($a \geqslant 0$ and $b > 0$). Following [6], we also define

$$K^2 \equiv \left(\frac{k}{2\pi T}\right)^2, \qquad M^2 \equiv \left(\frac{m}{2\pi T}\right)^2, \qquad \Omega^2 \equiv \left(\frac{\mu}{2\pi T}\right)^2$$

and obtain

$$J(M, a, b) = T(2\pi T)^{3+2a-2b} 2\Omega^{2\epsilon} \frac{\pi^{D/2}}{(2\pi)^D} \frac{\Gamma(D/2+a)}{\Gamma(D/2)} \frac{\Gamma(l)}{\Gamma(b)} S(M, l), \tag{9}$$

where

$$S(M, l) = \sum_{n=1}^{\infty} \frac{1}{(n^2 + M^2)^l}, \tag{10}$$

and $l = b - a - D/2$ and $D = 3 - 2\epsilon$. Depending upon the value of l, the series of equation (10) could be divergent and therefore need regularization.

Equation (10) is clearly of the form considered in equation (1) and one can write

$$S(M, l) = -\frac{1}{M^{2l}} + \zeta(2, l, M^2), \tag{11}$$

where

$$\zeta(2, l, M^2) = \frac{1}{M^{2l}} + \sum_{k=0}^{\infty} \frac{\Gamma(k+l)}{\Gamma(l)} \Psi_k(\lambda, 2, l, M^2) \tag{12}$$

and

$$\Psi_k(\lambda, 2, l, M^2) \equiv \frac{1}{(1+\lambda^2)^{s+k}} \sum_{j=0}^{k} \frac{(-M^2)^j}{j!(k-j)!} \lambda^{2(k-j)} \zeta(2(l+j)).$$

Once more we stress that although the series all converge to the same result independently of λ (provided that $\lambda > \frac{\xi-1}{2}$), the partial sums, obtained by truncating the series to a finite order, will necessarily display a dependence on the parameter. Such dependence, which is an artefact of working to a finite order, will also make the rate of convergence of the different elements of the family λ-dependent. Since it is desirable to obtain the most precise results with the least effort, one will select the optimally convergent series by fixing λ through the 'principle of minimal sensitivity' (PMS) [3]. The solutions obtained by applying this simple criterion display in general the highest convergence rate and, once plugged back in the original series, provide a *non-polynomial* expression in the natural parameters.

In the present case, the 'natural' parameter in equation (10) is M^2 and to first order the PMS yields

$$\lambda_{\text{PMS}}^{(1)} = M \sqrt{\frac{\zeta(2(1+l))}{\zeta(2l)}}. \tag{13}$$

On the other hand, if the value $\lambda = 0$ is chosen, then one obtains the series

$$\zeta(2, l, M^2) = \frac{1}{M^{2l}} + \sum_{k=0}^{\infty} \frac{\Gamma(k+l)}{\Gamma(l)} \frac{(-M^2)^k}{k!} \zeta(2(l+k)), \tag{14}$$

which, to a finite order yields a polynomial in M. Note that such series corresponds to the expansion used in [6].

In figure 1 we have compared $\zeta(2, 3/2, \xi)$ calculated numerically with the first-order approximations obtained by using equation (12) with $\lambda = \lambda_{\text{PMS}}^{(1)}$ and $\lambda = 0$, which are

$$\zeta_{\text{PMS}}^{(1)}(2, 3/2, \xi) = \frac{1}{\xi^{3/2}} + \frac{\zeta(3)^{5/2}}{(\zeta(3) + \xi\zeta(5))^{3/2}}, \tag{15}$$

$$\zeta_{\lambda=0}^{(1)}(2, 3/2, \xi) = \frac{1}{\xi^{3/2}} + \zeta(3) - \frac{3}{2}\xi\zeta(5), \tag{16}$$

where the second and third terms in the second equation can be obtained by Taylor expanding the first equation. In figure 2 we have plotted $\zeta(2, 3/2, \xi)$ as a function of $1/\xi$, showing that our series with few terms is very accurate even for $\xi \gg 1$.
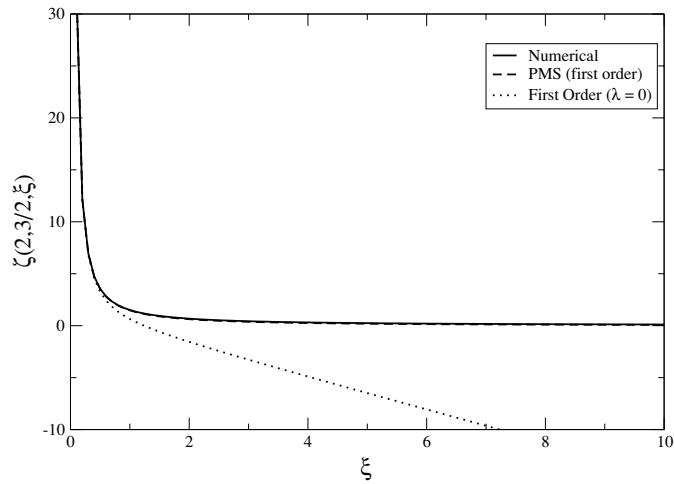
Figure 1. $\zeta(2, 3/2, \xi)$ calculated numerically (solid line), with the PMS to first order (dashed line) and using $\lambda = 0$ (dotted line).
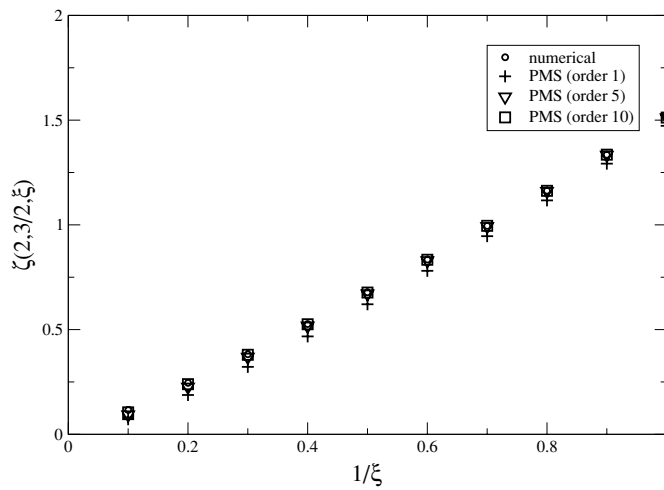

Figure 2. $\zeta(2, 3/2, \xi)$ calculated numerically (circles) and to order 1 (pluses), 5 (triangles) and 10 (squares) as a function of $1/\xi$, and using the λ_{PMS} to first order.

In table 1 we have calculated $\zeta(2, 3/2, 1)$ using equation (12) to order N with $\lambda_{\text{PMS}}^{(1)}$ (second column) and with $\lambda = 0$ (third column). Our formula, using the optimal parameter obtained to first order, converges exponentially to the exact value, whereas the formula corresponding to $\lambda = 0$ is actually useless (that formula is indeed limited to $\xi \ll 1$). Note that $\zeta(2, 3/2, 1)$ calculated with the first 10^5 terms in equation (1) gives the result 1.512 434 92150; the same precision is reached with our improved series with only 27 terms. We believe that this result by itself is sufficient to illustrate the strength of our method.

As an example of the application of our formula, we consider the case studied in equation (12) of [6], i.e.

Table 1. $\zeta(2, 3/2, 1)$ calculated using equation (12) to order N with $\lambda_{\text{PMS}}^{(1)}$ (second column) and with $\lambda = 0$ (third column). The underlined digits are correct.

N	$\zeta_{\text{PMS}}(2, 3/2, 1)$	$\zeta_{\lambda=0}(2, 3/2, 1)$
1	1.472 863 606 764 654 015 224 580 847 080 814 83	0.646 665 270 444 539 395 902 689 931 825 898 7386
10	1.512 414 154 851 640 213 060 386 995 257 442 65	3.405 482 727 150 825 241 793 479 671 674 666 0253
100	1.512 434 921 550 203 064 803 095 489 235 000 35	7.189 509 053 068 942 853 725 754 726 982 871 9810

$$I(m) = T\mu^{2\epsilon} \sum_{n=-\infty}^{+\infty} \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{k^2 + \omega_n^2 + M^2}$$

$$= T\mu^{2\epsilon} \int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{k^2 + M^2} + J(M, 0, 1), \tag{17}$$

which is essentially the one-loop self-energy.

The first integral can be calculated using dimensional regularization, i.e. by writing $\mathrm{d}^D k = \mathrm{d}\Omega_{D-1} k^{D-1} \, \mathrm{d}k$, letting $D = 3 - 2\epsilon$. The result obtained in this way is finite and equal to $-MT/(4\pi)$. The second term can be written as

$$J(M, 0, 1) = 2^{-2-2\epsilon} \pi^{-3/2-3\epsilon} \mu^{2\epsilon} T^{1-4\epsilon} \Gamma(-1/2 + \epsilon) \left[-\frac{1}{M^{-1-2\epsilon}} + \zeta(2, -1/2 + \epsilon, M^2) \right], \tag{18}$$

where the divergences are contained in $\zeta(2, -1/2 + \epsilon, M^2)$.

By using our equation (14) and retaining only the divergent terms and those independent of ϵ, we obtain

$$I(m) = -\frac{mT}{4\pi} + \frac{T^2}{12} - \frac{m^2}{16\pi^2} \left[\frac{1}{\epsilon} + \gamma - \log \frac{4\pi T^2}{\mu^2} \right] + \frac{m^4 \zeta(3)}{8(2\pi)^4 T^2}$$

$$- \frac{m^6 \zeta(5)}{1024\pi^6 T^4} + \frac{5m^8 \zeta(7)}{32\,768\pi^8 T^6} - \frac{7m^{10}\zeta(9)}{262\,144\pi^{10}T^8} + \cdots, \tag{19}$$

which reproduces the results of [6], which, however, were considered only up to order m^4. As anticipated, the formula obtained is a polynomial in m.

We will now use the optimal series (12) to improve this result. As a matter of fact, we can use the derivatives (integrals) of a series with respect to the parameter ξ,

$$\frac{\mathrm{d}\zeta(u, s, \xi)}{\mathrm{d}\xi} = -s\zeta(u, s + 1, \xi) \tag{20}$$

to analytically continue our results to any value of s. Indeed, by applying equation (20) twice, we can write the general expression

$$J(M, a, b) = 2^{1+2a-2b+2\epsilon} \Omega^{2\epsilon} \pi^{\frac{3}{2}+2a-2b+\epsilon} T^{4+2a-2b} \frac{\Gamma\left(\frac{3}{2} + a - \epsilon\right) \Gamma\left(-\left(\frac{3}{2}\right) - a + b + \epsilon\right)}{\Gamma(b)\Gamma\left(\frac{3}{2} - \epsilon\right)}$$

$$\times \left(1 + a - b + \frac{3 - 2\epsilon}{2}\right) \left(-a + b + \frac{-3 + 2\epsilon}{2}\right)$$

$$\times \left\{ \int \mathrm{d}M^2 \int \mathrm{d}M^2 \left(-M^{-2\left(2-a+b+\frac{-3+2\epsilon}{2}\right)} + \zeta\left(2, \frac{1}{2} - a + b + \epsilon, M^2\right)\right) \right\}, \tag{21}$$

where the last ζ function is now fully convergent when $a = 0$ and $b = 1$. Since this formula requires a double integration of a convergent series, the result is determined up to two constants of integration, which will contain the divergent contributions.

By applying equation (12) to first order and using the optimal value of λ given by equation (13), we obtain the simple result

$$J_{\text{PMS}}(M, 0, 1) = T^2 \left(-\kappa_1 M^2 - \frac{\zeta(3)^{5/2}\sqrt{\zeta(3) + M^2\zeta(5)}}{\zeta(5)^2} \right) + \kappa_2, \tag{22}$$

where $\kappa_{1,2}$ are the constants of integration *independent* of M. In order to determine these constants, we Taylor expand this expression in M and then use $M = m/(2\pi T)$, thus obtaining

$$J(M, 0, 1) \approx \kappa_2 - \frac{(\kappa_1 m^2)}{4\pi^2} + \frac{m^4 \zeta(3)}{128\pi^4 T^2} - \frac{T^2 \zeta(3)^3}{\zeta(5)^2} - \frac{m^2 \zeta(3)^2}{8\pi^2 \zeta(5)} - \frac{m^6 \zeta(5)}{1024\pi^6 T^4}$$
$$+ \frac{5m^8 \zeta(5)^2}{32\,768\pi^8 T^6 \zeta(3)} - \frac{7m^{10}\zeta(5)^3}{262\,144\pi^{10}T^8\zeta(3)^2} + \cdots. \tag{23}$$

This expression can now be compared with the perturbative result and the constants of integration can thus be extracted:

$$\kappa_1 = \frac{1}{4}\left[\frac{1}{\epsilon} + \gamma - \log\left(\frac{4\pi T^2}{\mu^2}\right) - \frac{2\zeta(3)^2}{\zeta(5)}\right],$$

$$\kappa_2 = \frac{1}{12} + \frac{\zeta(3)^3}{\zeta(5)^2}.$$

Equation (22) is a quite remarkable formula: indeed, although it has been derived by applying our method to first order, it reproduces correctly the terms going as m^4 and m^6, and it also provides the coefficients of the higher order terms apart from factors depending on the Riemann ζ function evaluated at odd integer values[1]. Of course, we can let the perturbative result guide us further and use the fact that $\lim_{n\widetilde{\Omega}\infty} \zeta(n) = 1$ to write the improved formula

$$J_{\text{PMS2}}(M, 0, 1) = \frac{13}{12}T^2 - \frac{M^2 T^2}{4}\left[\frac{1}{\epsilon} + \gamma - \log\left(\frac{4\pi T^2}{\mu^2}\right) - 2\right]$$
$$- \frac{T^2}{16}\left[\left(16\sqrt{1 + M^2} - 2M^4(-1 + \zeta(3)) + M^6(-1 + \zeta(5))\right)\right], \tag{24}$$

where the notation PMS2 has been introduced to distinguish it from the previous formula. Note that both equations (22) and (24) are non-polynomial in m.

By putting the pieces together, we finally obtain our approximation to $I(m)$, given by

$$I_{\text{PMS2}}(M) = -\frac{MT^2}{2} + \frac{13}{12}T^2 - \frac{M^2 T^2}{4}\left[\frac{1}{\epsilon} + \gamma - \log\left(\frac{4\pi T^2}{\mu^2}\right) - 2\right]$$
$$- \frac{T^2}{16}\left[\left(16\sqrt{1 + M^2} - 2M^4(-1 + \zeta(3)) + M^6(-1 + \zeta(5))\right)\right]. \tag{25}$$

When negative values of m^2 are considered, corresponding to a spontaneously broken phase, these results hold for $T \geqslant T_0 \equiv -m^2/(4\pi^2)$.

As a further application of the method described in this paper, we consider the integral

$$\mathcal{J}(\phi) = \frac{T}{2}\mu^{-2\epsilon}\sum_{k_0}\int\frac{\mathrm{d}^{3-2\epsilon}k}{(2\pi)^{3-2\epsilon}}\log(k^2 + m^2(\phi)), \tag{26}$$

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[1] We have checked this property to much higher orders than the ones reached in the formula.

where $k_0 = 2\pi nT$, for all n integers. The effective potential to one loop is expressed in terms of this integral.

Given the relation

$$I(m) = \frac{1}{m}\frac{\mathrm{d}\mathcal{J}}{\mathrm{d}m}, \tag{27}$$

it is not necessary to calculate $\mathcal{J}(m)$ which can be obtained as

$$\mathcal{J}(m) = \int I(m)m\,\mathrm{d}m + \rho, \tag{28}$$

where ρ is a constant of integration independent of m.

By using I_{PMS2} of equation (25), we obtain the simple expression

$$\mathcal{J}_{\mathrm{PMS2}}(m) = \rho + \frac{13}{24}m^2T^2 - \frac{m^3T}{12\pi} - \frac{T}{6\pi}(m^2 + 4\pi^2T^2)^{3/2}$$
$$- \frac{m^4}{64\pi^2}\left[\frac{1}{\epsilon} + \gamma - \log\left(\frac{4\pi T^2}{\mu^2}\right) - 2\right] + \frac{m^6}{768\pi^4T^2}(\zeta(3) - 1)$$
$$- \frac{m^8}{8192\pi^6T^4}(\zeta(5) - 1), \tag{29}$$

which can be expanded in powers of m to give

$$\mathcal{J}_{\mathrm{PMS2}}(m) = \rho - \frac{4\pi^2T^4}{3} + \frac{m^2T^2}{24} - \frac{m^3T}{12\pi} - \frac{m^4}{64\pi^2}\left(\frac{1}{\epsilon} + \gamma - \log\left(\frac{4\pi T^2}{\mu^2}\right)\right)$$
$$+ \frac{m^6\zeta(3)}{768\pi^4T^2} - \frac{m^8\zeta(5)}{8192\pi^6T^4} + \frac{m^{10}}{65\,536\pi^8T^6} + \cdots, \tag{30}$$

which reproduces exactly equation (3.7) of [7], provided that $\rho = \frac{4\pi^2T^4}{3}$ + constant. This last constant is independent both of m and of T.

We are now in a position to compare our result of equation (29) with the high-temperature expansion of equation (42) of [8] which reads[2]

$$\frac{\Omega}{2VT^4} = -\frac{\pi^2}{90} + \frac{y^2}{6} - \frac{y^3}{12\pi} + \frac{y^4}{32\pi^2}\left[\log\frac{4\pi}{y} - \gamma + \frac{3}{4}\right]$$
$$- \frac{y^4}{16\pi^2}\sum_{k=1}^{\infty}(-1)^k\left(\frac{y}{4\pi}\right)^k\frac{\Gamma(2k+1)\zeta(2k+1)}{\Gamma(k+1)\Gamma(k+3)}, \tag{31}$$

after setting the chemical potential to zero[3]. We have defined $y = m/T$.

As we can see from table 2, the result of [6] for $I(m)$ would essentially provide the expansion of equation (31) and thus reproduce the results obtained long time ago by Haber and Weldon [8]. We regard this procedure as 'perturbative' meaning that, when the series in equation (31) is truncated, a polynomial in powers of m/T is obtained. On the other hand, our simple formula, equation (29), reproduces correctly the perturbative expansion of [8] up to order $(m/T)^8$, and up to a factor involving the ζ function calculated at odd integer values, which however tends to 1 for large values. This makes our simple formula quite precise. We remind the reader that equation (29) was obtained by applying our method to *first order* and that, given the convergence of the method, drastic improvements in the quality of the approximation are expected if higher orders would to be taken into account. Although the calculation of higher orders with our method would be an interesting issue by itself, we will leave it for future work.

[2] We divide by an overall factor of 2.
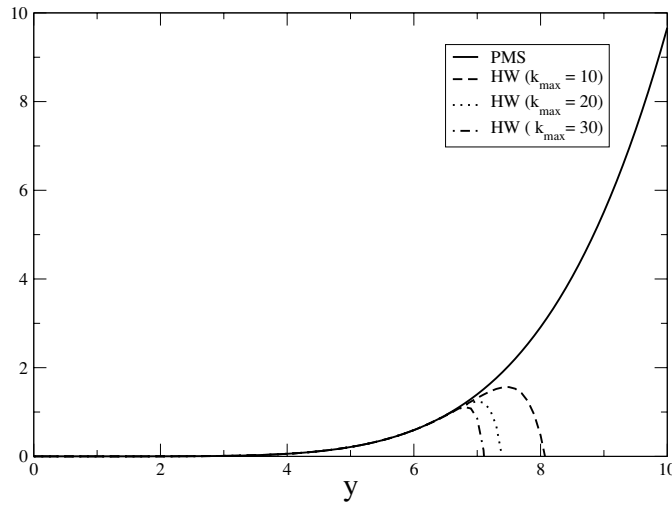[3] In such limit, the hypergeometric functions in the original formula of [8] all go to 1.

Figure 3. Comparison between $\tilde{\mathcal{J}}_{\mathrm{PMS2}}/T^4$ and equation (31) taking $k_{\max} = 10$ (dashed), $k_{\max} = 20$ (dotted) and $k_{\max} = 30$ (dot-dashed) terms.

Table 2. Comparison between equation (31) and the simple approximation of equation (29).

N	Equation (31)	$\mathcal{J}_{\mathrm{PMS2}}/T^4$
$\left[\frac{m}{T}\right]^6$	$\frac{\zeta(3)}{768\pi^4}$	$\frac{\zeta(3)}{768\pi^4}$
$\left[\frac{m}{T}\right]^8$	$-\frac{\zeta(5)}{8192\pi^6}$	$-\frac{\zeta(5)}{8192\pi^6}$
$\left[\frac{m}{T}\right]^{10}$	$\frac{\zeta(7)}{65\,536\pi^8}$	$\frac{1}{65\,536\pi^8}$
$\left[\frac{m}{T}\right]^{12}$	$-\frac{7\zeta(9)}{3145\,728\pi^{10}}$	$-\frac{7}{3145\,728\pi^{10}}$
$\left[\frac{m}{T}\right]^{14}$	$\frac{3\zeta(11)}{8388\,608\pi^{12}}$	$\frac{3}{8388\,608\pi^{12}}$

In figure 3 we have plotted $\tilde{\mathcal{J}}_{\mathrm{PMS2}}/T^4$, defined by taking out the terms up to order $(m/T)^4$, and the similar functions obtained from equation (31), performing the sum to three different orders ($k_{\max} = 10, 20, 30$). The horizontal scale is $y = m/T$. We leave to the reader any judgement on the quality of our approximation. Note that the series of Haber and Weldon should correspond to our variational series for $\lambda = 0$; as we have seen before the criterion of convergence for a series is that $\lambda^2 > \frac{M^2-1}{2}$, which for $\lambda = 0$ can be fulfilled only if $m/T < 2\pi$. Looking at the figure we indeed see that when we get close to this value, the sum of [8] becomes ill-behaved.

4. Conclusions

The method that we have described in this paper is quite general and probably could be extended in the future to deal with a larger class of problems than the one presented here. We have proved that, by using variational techniques, it is possible to estimate analytically and to an arbitrary degree of precision series which are difficult to evaluate with standard techniques. In particular, we have shown that by relating divergent series to convergent ones through repeated derivatives and then matching the divergences contained in the constants of integration with the ones coming out of the perturbative calculation, it is possible to obtain a

non-polynomial expression in the inverse temperature. The expressions obtained in this way are very accurate even at low orders and low temperatures. There is a huge literature dealing with field theoretical problems at finite temperature and we feel that this paper can provide a quite general and useful tool to attack many of these problems. We also believe, although at present it is still to be confirmed, that the method that we have described could be useful in the non-perturbative calculation of the Casimir effect, for which zeta function regularization is a well-established technique (see for example [9]). It would be quite interesting to see if non-perturbative expressions for the Casimir effect could be calculated analytically. We hope to apply these ideas to such a problem in the near future. As a final remark, we would like to stress that despite the simplicity of the ideas that we have illustrated, all the results that we have obtained are *fully analytical* and improvable to the desired level of accuracy.

Acknowledgments

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